Pairs of Cauchy Singular Integral Equations and the Kernel $[b(z) + a(\zeta)]/(z-\zeta)$

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PAIRS OF CAUCHY SINGULAR INTEGRAL EQUATIONS
AND THE KERNEL \([b(z) + a(\zeta)]/(z-\zeta)\)

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Abstract

This report presents closed formulas for the solution of

\[ a(\zeta) \int_{L}^{\phi_1(z)dz} + b(z) \int_{L}^{\phi_1(z)dz} = f_1(\zeta) \]

provided the coefficients satisfy

\[ [s(\zeta) - a(\zeta)][s(\zeta) - b(\zeta)] = [(\zeta - \alpha)(\zeta - \beta)]^\kappa t^2(\zeta) \]

where \( s(\zeta) \) and \( t(\zeta) \) are rational functions and \( \kappa \) is either zero or one.
1. Introduction

The purpose of this paper is to review briefly the theory of pairs of Cauchy singular integral equations, namely,

\[(1.1) \sum_{j=1}^{2} \left[ a_{ij}(\zeta) \phi_j(\zeta) + b_{ij}(\zeta) \int_{L} \frac{\phi_j(z)dz}{z-\zeta} \right] = f_i(\zeta), \quad i = 1, 2;\]

with the object of finding new cases for which the solution of (1.1) can be presented in closed form, that is, in terms of a finite number of quadratures. The attainment of this objective is connected with the solution of a nonlinear barrier equation.

Vekua [1] has demonstrated that a closed solution formula for (1.1) exists if the \(a_{ij}(\zeta)\) and \(b_{ij}(\zeta)\) are rational functions and if \(L\) is taken to be a closed curve \(C\). MacCamy [2] has shown that an extension of a method used to solve a single equation yields the solution of (1.1) in closed form provided the coefficients are constants.

The equation

\[(1.2) \quad a(\zeta) \int_{L} \frac{\phi_1(z)dz}{z-\zeta} + \int_{L} \frac{b(z)\phi_1(z)dz}{z-\zeta} = f_1(\zeta)\]

has been studied by several authors. If either \(a(\zeta)\) or \(b(\zeta)\) is a rational function then (1.2) can be reduced to the well-known equation

\[(1.3) \quad \int_{L} \frac{\omega(z)dz}{z-\zeta} = \gamma(\zeta).\]
Aside from this case, Lewin [3] has shown that

\[(1.4) \quad a(\zeta) \int\limits_{-1}^{1} \frac{\phi_1(z)dz}{z-\zeta} + \int\limits_{-1}^{1} \frac{b(z)\phi_1(z)dz}{z-\zeta} = f_1(\zeta)\]

can be solved in closed form if

\[a(\zeta) = b(\zeta)\]

or

\[a(\zeta)b(\zeta) = (1-\zeta^2)s^2(\zeta)\]

where \(s(\zeta)\) is a rational function. Using a method different from Lewin's, Bittner and Hirche [4] have shown how to solve (1.2) for a general path \(L\) when \(a(\zeta) = b(\zeta)\).

If we introduce

\[(1.5) \quad b(\zeta)\phi_1(\zeta) = \phi_2(\zeta)\]

then (1.2) becomes

\[(1.6) \quad a(\zeta) \int\limits_L \frac{\phi_1(z)dz}{z-\zeta} + \int\limits_L \frac{\phi_2(z)dz}{z-\zeta} = f_1(\zeta)\]

and hence (1.6) and (1.5) constitute a system of the type (1.1). In the sequel we present closed solution formulas for (1.2) when \(a(\zeta)\) and \(b(\zeta)\) are so related that

\[(1.7) \quad [s(\zeta) - a(\zeta)][s(\zeta) - b(\zeta)] = [((\zeta-\alpha)(\zeta-\beta))]^{\kappa}t^2(\zeta).\]

Here \(\kappa\) is zero or one; the functions \(s(\zeta)\) and \(t(\zeta)\) are assumed to be
rational functions; while $\alpha$ and $\beta$ are the end points of the path $L$. The relation (1.7) embraces and extends the known results mentioned above. We also show in the final section that if $L = C$ where $C$ is a smooth closed path, then (1.1) can be solved in an elementary manner if the coefficients satisfy certain analyticity conditions.

2. Theory of a Pair of Cauchy Singular Integral Equations

The functions $a_{ij}(\zeta)$, $b_{ij}(\zeta)$, $\phi_j(\zeta)$ and $f_i(\zeta)$ which appear in the equations

$$(2.1) \quad \sum_{j=1}^{2} \left[ a_{ij}(\zeta) \phi_j(\zeta) + b_{ij}(\zeta) \int_{L}^{\zeta} \frac{\phi_j(z)dz}{z-\zeta} \right] = f_i(\zeta), \quad i = 1, 2$$

can be regarded as defining the vector functions

$$\phi(\zeta) = \begin{pmatrix} \phi_1(\zeta) \\ \phi_2(\zeta) \end{pmatrix}, \quad f(\zeta) = \begin{pmatrix} f_1(\zeta) \\ f_2(\zeta) \end{pmatrix}$$

and the matrix functions

$$A(\zeta) = \begin{pmatrix} a_{11}(\zeta) & a_{12}(\zeta) \\ a_{21}(\zeta) & a_{22}(\zeta) \end{pmatrix}, \quad B(\zeta) = \begin{pmatrix} b_{11}(\zeta) & b_{12}(\zeta) \\ b_{21}(\zeta) & b_{22}(\zeta) \end{pmatrix}.$$ 

By definition, a vector or matrix function is said to have a certain property if and only if each of its components, or elements, possesses the property.
In terms of the above functions, the system (2.1) can be written in the matrix form

\[(2.2) \quad A(\zeta)\phi(\zeta) + B(\zeta) \int_{L} \frac{\phi(z)dz}{z - \zeta} = f(\zeta) .\]

In (2.2) the function \(\phi(z) = \phi(x + iy)\) is supposed to be a complex function which is integrable along the simple, smooth, directed curve \(L\). If \(\alpha\) and \(\beta \neq \alpha\) are respectively the initial and terminal points of \(L\), then \(\zeta\) is a point in \(L' = L - \alpha - \beta\). Such a point can be imbedded in a subarc \(L_\varepsilon\) of \(L\) such that the extremities of \(L_\varepsilon\) are the same distance \(\varepsilon\) from \(\zeta\). The integral symbol in (2.2) means the Cauchy principal value, that is,

\[(2.3) \quad \int_{L} \frac{\phi(z)dz}{z - \zeta} = \lim_{\varepsilon \to 0} \int_{L - L_\varepsilon} \frac{\phi(z)dz}{z - \zeta}\]

provided the limit exists. Although it is not a necessary condition, we assume for simplicity that each component which is introduced in (2.2) is Hölder continuous in \(L'\). This insures the existence of (2.3) and the principal values of other integrals in the sequel.

With respect to the behavior of functions at the end points of \(L\) we admit singularities there to the extent that they are integrable.

If a solution of (2.2) exists we can introduce the vector function

\[(2.4) \quad \chi(w) = \int_{L} \frac{\phi(z)dz}{z - w} .\]
This function is analytic in the complex $w$-plane slit along $L$, and it vanishes as $w \to \infty$. If the unit tangent vector to the path $L$ is rotated counterclockwise $\pi/2$ radians, we obtain the unit normal which by definition points toward the positive side of $L$. If $w$ is allowed to approach $\zeta$ from the positive side of $L$ and then from the negative side, the Plemelj formulas show that the limit values of $\chi(w)$ must satisfy

$$\chi^+(\zeta) = \pi i \phi(\zeta) + \int_L \frac{\phi(z)dz}{z-\zeta}$$

$$\chi^-(\zeta) = -\pi i \phi(\zeta) + \int_L \frac{\phi(z)dz}{z-\zeta}$$

from which

$$2\pi i \phi(\zeta) = \chi^+(\zeta) - \chi^-(\zeta) \quad (2.5)$$

and

$$2 \int_L \frac{\phi(z)dz}{z-\zeta} = \chi^+(\zeta) + \chi^-(\zeta) \quad (2.6)$$

With (2.5) and (2.6) we see that equation (2.2) can be written as

$$[A(\zeta) + \pi i B(\zeta)]\chi^+(\zeta) - [A(\zeta) - \pi i B(\zeta)]\chi^-(\zeta) = 2\pi i f(\zeta) \quad (2.7)$$

Let

$$C(\zeta) = A(\zeta) + \pi i B(\zeta)$$

$$D(\zeta) = A(\zeta) - \pi i B(\zeta)$$

and suppose that the inverses $C^{-1}(\zeta)$ and $D^{-1}(\zeta)$ exist. Also let
\[ E(\zeta) = C^{-1}(\zeta)D(\zeta) \]
\[ g(\zeta) = C^{-1}(\zeta)f(\zeta) . \]

In terms of these functions equation (2.7) is

\[ \chi^+(\zeta) - E(\zeta)\chi^-(\zeta) = 2\pi ig(\zeta) . \]  

This equation, involving the limit values of \( \chi(w) \) for \( \zeta \) in \( L' \), will be referred to as the barrier equation for the determination of \( \chi(w) \). If this equation is satisfied by a function \( \chi(w) \) with the properties noted above, then a solution of (2.2) can presumably be recovered from \( \chi(w) \) by using (2.5).

The solution of (2.8) depends on the solution of the homogeneous barrier equation

\[ M^+(\zeta) = E(\zeta)M^-(\zeta) \]

where \( M(w) \) is a matrix function and \( E^{-1}(\zeta) \) exists for each \( \zeta \) on \( L' \). The analysis of (2.9) can be reduced to the analysis of a system of Fredholm equations, see [1]. It can be shown that there exists a matrix function \( R(w) \) with components \( r_{ij}(w) \) such that:

1. \( R(w) \) satisfies

\[ R^+(\zeta) = E(\zeta)R^-(\zeta) \]

for \( \zeta \) in \( L' = L - \alpha - \beta \).

2. Each component \( r_{ij}(\zeta) \) is Hölder continuous for \( \zeta \) in \( L' \).
3. The inverse of $R(w)$

\[(2.11) \quad R^{-1}(w) = (r^*_{ij}) = \frac{1}{r_{11}r_{22} - r_{12}r_{21}} \begin{pmatrix} r_{22} & -r_{12} \\ -r_{21} & r_{11} \end{pmatrix}\]

is analytic in the finite slit plane $\Lambda$ which is the entire complex plane minus $L$ and the point at infinity. $R^{-1}(\zeta)$ and the limit values $[R^+(\zeta)]^{-1}$, $[R^-(\zeta)]^{-1}$ are Hölder continuous on $L$.

4. As $w$ approaches infinity $r^*_{ij}(w)$ behaves like $c_{ij}w^{n_{ij}}$ where each $n_{ij}$ may be a positive or negative integer or zero. The maximum of $n_{ij}$, say $n$, is called the degree of $R^{-1}(w)$ at infinity.

Once the existence of $R^{-1}(w)$ has been established the equation (2.8) can be multiplied by $[R^+(\zeta)]^{-1}$ to give

\[ [R^+(\zeta)]^{-1} \chi^+(\zeta) - [R^+(\zeta)]^{-1} E(\zeta) \chi^-(\zeta) = 2\pi i [R^+(\zeta)]^{-1} g(\zeta). \]

This, by virtue of (2.10), is the same as

\[(2.12) \quad [R^+(\zeta)]^{-1} \chi^+(\zeta) - [R^-(\zeta)]^{-1} \chi^-(\zeta) = 2\pi i [R^+(\zeta)]^{-1} g(\zeta). \]

The Plemelj formulas show that the general solution of (2.12) is

\[(2.13) \quad [R(w)]^{-1} \chi(w) = \int_L \frac{[R^+(z)]^{-1} g(z) \, dz}{z-w} + p(w) \]

where $p(w)$ is a vector function analytic in $\Lambda$ and such that

\[ p^+(\zeta) = p^-(\zeta). \]
Since $R^{-1}(w)$ is of finite degree at infinity and is bounded in the neighborhoods of the end points of $L$; and since our functions are limited to integrable singularities at the end points, it follows that each component of $p(w)$ is a polynomial of degree no greater than $n-1$ if $n > 0$. If $n \leq 0$, then $p(w) = 0$.

From (2.13) we have

\begin{equation}
\chi(w) = R(w) \left[ \int_{L} \frac{[R^+(z)]^{-1}g(z)dz}{z-w} + p(w) \right].
\end{equation}

The properties of the right-hand side of (2.14) must match those of $\chi(w)$. This means that in order for a solution of (2.2) to exist it may be necessary to impose extra conditions on $g(z)$ and the vector polynomial $p(w)$ so that the right-hand side of (2.14) is analytic in $\Lambda$ and vanishes like $c/w$ as $w \to \infty$. For example, if the degree of $R^{-1}(w)$ at infinity is $n = -k$ where $k > 0$, then $p(w) = 0$; and if $R(w)$ is analytic in $\Lambda$ with degree $k$ at infinity then

$$R(w) \int_{L} \frac{[R^+(z)]^{-1}g(z)dz}{z-w}$$

behaves in the same way as $\chi(w)$ does when $w$ approaches infinity if and only if $g(z)$ is such that

$$\int_{L} [R^+(z)]^{-1}g(z)z^j dz = 0 , \quad j = 0, 1, \ldots, k-1 .$$

Again if $R(w)$ has a simple pole at $w = w_1$ then $g(z)$ and $p(w)$ must be such that
\[
\int \frac{[R^+(z)]^{-1}g(z)}{z-w_1} \, dz + p(w_1) = 0. 
\]

From (2.14) the difference of the limit values of \(\chi(w)\) is

\[
\chi^+(\zeta) - \chi^-(\zeta) = \pi i \{ I + R^-(\zeta)[R^+(\zeta)]^{-1} \} g(\zeta) 
+ \{ R^+(\zeta) - R^-(\zeta) \} \left\{ \int \frac{[R^+(z)]^{-1}g(z)}{z-\zeta} \, dz + p(\zeta) \right\}
\]

where \(I\) is the unit matrix. Since

\[
R^-(\zeta)[R^+(\zeta)]^{-1} = E^{-1}(\zeta) = D^{-1}(\zeta)C(\zeta)
\]

and

\[
g(z) = C^{-1}(z)f(z),
\]

the difference can be written

\[
\chi^+(\zeta) - \chi^-(\zeta) = \pi i \{ C^{-1}(\zeta) + D^{-1}(\zeta) \} f(\zeta) 
+ \{ R^+(\zeta) - R^-(\zeta) \} \left\{ \int \frac{[R^+(z)]^{-1}C^{-1}(z)f(z)}{z-\zeta} \, dz + p(\zeta) \right\}.
\]

The sum of the limit values is

\[
\chi^+(\zeta) - \chi^-(\zeta) = \pi i \{ C^{-1}(\zeta) - D^{-1}(\zeta) \} f(\zeta) 
+ \{ R^+(\zeta) + R^-(\zeta) \} \left\{ \int \frac{[R^+(z)]^{-1}C^{-1}(z)f(z)}{z-\zeta} \, dz + p(\zeta) \right\}.
\]

That

\[
\phi(z) = \frac{1}{2\pi i} \left[ \chi^+(z) - \chi^-(z) \right]
\]
actually satisfies the integral equation (2.2) can be seen by applying the extended Cauchy integral formula to the domain \( \Lambda \). We have

\[
\int_{L} \frac{\phi(z)dz}{z-\zeta} = \frac{1}{2\pi i} \int_{L} \frac{[\chi^+(z) - \chi^-(z)]dz}{z-\zeta}
\]

\[
= \frac{1}{2\pi i} \int_{L^+ - L^-} \frac{\chi(z)dz}{z-\zeta}
\]

where \( L^+ - L^- \) denotes the clockwise path along the positive and negative banks of \( L \). Then, since \( \chi(w) \) is analytic in \( \Lambda \) and vanishes at infinity the deformation of \( L^+ - L^- \) into a circle centered at the origin with radius as large as we please produces

\[
\int_{L} \frac{\phi(z)dz}{z-\zeta} = \frac{1}{2} \left[ \chi^+(\zeta) + \chi^-(\zeta) \right]
\]

\[
= \frac{\pi i}{2} \left[ C^{-1}(\zeta) - D^{-1}(\zeta) \right] f(\zeta)
\]

\[
+ \frac{1}{2} \left\{ R^+(\zeta) + R^-(\zeta) \right\} \left( \int_{L} \frac{[R^+(z)]^{-1}C^{-1}(z)f(z)dz}{z-\zeta} + p(\zeta) \right)
\]

Hence

\[
A(\zeta)\Phi(\zeta) + B(\zeta) \int_{L} \frac{\phi(z)dz}{z-\zeta}
\]

\[
= \frac{1}{2} \left\{ A(\zeta)[C^{-1}(\zeta) + D^{-1}(\zeta)] + 2\pi i B(\zeta)[C^{-1}(\zeta) - D^{-1}(\zeta)] \right\} f(\zeta)
\]

\[
+ \frac{1}{2} \left( \frac{A(\zeta)}{\pi i} \left[ R^+(\zeta) - R^-(\zeta) \right] + B(\zeta)[R^+(\zeta) + R^-(\zeta)] \right)\left( \int_{L} \frac{[R^+(z)]^{-1}C^{-1}(z)f(z)dz}{z-\zeta} + p(\zeta) \right)
\]

10
The right-hand side of this equation is equal to \( f(\zeta) \) because

\[
A(\zeta) + \pi i B(\zeta) = C(\zeta)
\]

\[
A(\zeta) - \pi i B(\zeta) = D(\zeta)
\]

and

\[
[A(\zeta) + \pi i B(\zeta)]R^+(\zeta) = [A(\zeta) - \pi i B(\zeta)]R^-(\zeta).
\]

We have now shown that if the solution \( R(w) \) of (2.9) can be expressed in terms of a finite number of quadratures and if a solution of

\[
(2.15) \quad A(\zeta)\phi(\zeta) + B(\zeta) \int_{\mathcal{L}} \frac{\phi(z)dz}{z-\zeta} = f(\zeta)
\]

exists, then

\[
(2.16) \quad \phi(\zeta) = \frac{1}{2} \left[ C^{-1}(\zeta) + D^{-1}(\zeta) \right] f(\zeta)
\]

\[
+ \frac{1}{2\pi i} \left( R^+(\zeta) - R^-(\zeta) \right) \int_{\mathcal{L}} \frac{[R^+(z)]^{-1}C^{-1}(z)f(z)dz}{z-\zeta} + p(\zeta)
\]

is the closed solution of (2.15). Each component of the vector function \( p(\zeta) \) is a polynomial of degree not exceeding \( n-1 \) if the degree of \( R^{-1}(w) \) at infinity is \( +n \) where \( n > 0 \). If \( -n = k > 0 \), then \( p(\zeta) = 0 \). If (2.15) is soluble, then \( g(z) = C^{-1}(z)f(z) \) and \( p(w) \) must be such that

\[
R(w) \int_{\mathcal{L}} \frac{[R^+(z)]^{-1}g(z)dz}{z-w} + p(w)
\]

is analytic in \( \Lambda \), and vanishes like \( c/w \) as \( w \) approaches infinity.
3. The Homogeneous Barrier Equation

The analysis in Section 2 shows that the basic problem with respect to the system of integral equation (2.1), i.e.,

\[(3.1) \quad A(\xi)\phi(\xi) + B(\xi) \int_{L} \frac{\phi(z)dz}{z-\xi} = f(\xi)\]

is the problem of finding an appropriate matrix function

\[
R(w) = \begin{pmatrix} r_{11}(w) & r_{12}(w) \\ r_{21}(w) & r_{22}(w) \end{pmatrix} = (r_{ij}(w))
\]

which satisfies the homogeneous barrier equation

\[(3.2) \quad R^+(\xi) = E(\xi)R^-(\xi)\]

where

\[(3.3) \quad E(\xi) = C^{-1}(\xi)D(\xi)\]

and

\[(3.4) \quad C(\xi) = A(\xi) + \pi i B(\xi); \quad D(\xi) = A(\xi) - \pi i B(\xi).\]

If the quantities in (3.2) were scalars, as they are for the case of a single equation, we could solve (3.2) by taking the logarithm of each side of the equation. If \(M\) is a matrix function we can define \(e^M\) in terms of \(M\) and hence we can define \(\ln M\). However, \(e^{M+N} = e^M \cdot e^N\) is true if and only if the matrices \(M\) and \(N\) commute, that is \(MN = NM\). This implies that we cannot always replace \(\ln MN\) with \(\ln M + \ln N + n2\pi i\).
Therefore it is only for exceptional cases; for example the case in which $E$ is a constant matrix, that we can find solutions of (3.2) by solving

$$\ln R^+(\zeta) - \ln R^-(\zeta) = \ln E(\zeta).$$

The problem of solving (3.2) is equivalent to the problem of finding column vectors

$$(3.5) \quad r_1(w) = \begin{pmatrix} r_{11}(w) \\ r_{21}(w) \end{pmatrix}; \quad r_2(w) = \begin{pmatrix} r_{12}(w) \\ r_{22}(w) \end{pmatrix}$$

such that

$$r_1^+(\zeta) = E(\zeta)r_1^-(\zeta)$$

and

$$r_2^+(\zeta) = E(\zeta)r_2^-(\zeta).$$

This means that if

$$(3.6) \quad \psi(w) = \begin{pmatrix} \psi_1(w) \\ \psi_2(w) \end{pmatrix}$$

is a vector function, then we can construct $R(w)$ from appropriate solutions of

$$(3.7) \quad \psi^+(\zeta) = E(\zeta)\psi^-(\zeta)$$

if such solutions exist. The analysis of (3.7) can be reduced to the analysis of a system of quasi-regular Fredholm equations. See [1]. Of course this reduction is of basic theoretical importance but it is
not very helpful when we are seeking closed solutions of (3.7). For this reason we intend to avoid transforming (3.7) into a matrix integral equation but concentrate on (3.7) and its component equations.

If we introduce the components of \( E = (e_{ij}(\zeta)) \) which are defined by

\[
(3.8) \quad C^{-1}(\zeta)D(\zeta) = E(\zeta) = \begin{pmatrix} e_{11}(\zeta) & e_{12}(\zeta) \\ e_{21}(\zeta) & e_{22}(\zeta) \end{pmatrix},
\]

the equation (3.7) is equivalent to the system

\[
(3.9) \quad \begin{align*}
\psi_1^+(\zeta) &= e_{11}(\zeta)\psi_1^-(\zeta) + e_{12}(\zeta)\psi_2^-(\zeta) \\
\psi_2^+(\zeta) &= e_{21}(\zeta)\psi_1^-(\zeta) + e_{22}(\zeta)\psi_2^-(\zeta).
\end{align*}
\]

In some cases the components \( \psi_1 \) and \( \psi_2 \) can be obtained in closed form from (3.9) because the coefficients are such that the system can be reduced to the scalar equation

\[
\omega^+(\zeta) - \omega^-(\zeta) = \lambda(\zeta)
\]

where \( \omega(w) \) is a function of \( \psi_1 \) and \( \psi_2 \). It is also to be noted that if we set

\[
(3.10) \quad \psi_1(w) = \ell(w)\psi_2(w)
\]

where \( \ell(w) \) is analytic in \( \Lambda \), then the system (3.9) can be reduced to a single barrier equation for \( \ell(w) \). The substitution of (3.10) in (3.9) shows that \( \ell(w) \) must satisfy
(3.11) $e_{21}(\zeta)\ell^+(\zeta)\ell^-(\zeta) + e_{22}(\zeta)\ell^+(\zeta) - e_{11}(\zeta)\ell^-(\zeta) - e_{12}(\zeta) = 0$.

This is a nonlinear barrier equation but it may be possible to linearize it after performing certain manipulations on $\ell(w)$. For example, if $E$ is a constant matrix and if we let

$$\ell(w) = \frac{1}{\ell(w)} + k$$

where the constant $k$ satisfies

$$e_{21}k^2 + (e_{22} - e_{11})k - e_{12} = 0$$

then (3.11) becomes

(3.12) $e_{21} + (ke_{21} + e_{22})\ell^-(\zeta) + (ke_{21} - e_{11})\ell^+(\zeta) = 0$,

a linear barrier equation which can easily be solved for $\ell(w)$. The next section presents other illustrations.

4. An Integral Equation with the Kernel $[a(\zeta) + b(z)]/(z - \zeta)$

subject to $[s(\zeta) - a(\zeta)][s(\zeta) + b(\zeta)] = (\zeta - \alpha)(\zeta - \beta)t^2(\zeta)$

The equation

(4.1) $a(\zeta)\int_{L}^{\phi_1(z)dz} + \int_{L}^{b(z)\phi_1(z)dz} = f_1(\zeta)$

is equivalent to the system
\[
\begin{aligned}
&\left\{ \begin{array}{l}
a(\zeta) \int_{\mathbb{L}} \frac{\phi_1(z) dz}{z-\zeta} + \int_{\mathbb{L}} \frac{\phi_2(z) dz}{z-\zeta} = f_1(\zeta) \\
 b(\zeta)\phi_1(\zeta) - \phi_2(\zeta) = 0 \\
\end{array} \right.
\end{aligned}
\]

(4.2)

This is the particular case of (2.2), namely,

(4.3)

\[
A(\zeta)\phi(\zeta) + B(\zeta) \int_{\mathbb{L}} \frac{\phi(z) dz}{z-\zeta} = f(\zeta)
\]

which appears if we take

\[
\phi(\zeta) = \begin{pmatrix} \phi_1(\zeta) \\ \phi_2(\zeta) \end{pmatrix}, \quad f(\zeta) = \begin{pmatrix} f_1(\zeta) \\ 0 \end{pmatrix},
\]

(4.4)

\[
A(\zeta) = \begin{pmatrix} 0 & 0 \\ b(\zeta) & -1 \end{pmatrix}, \quad B(\zeta) = \begin{pmatrix} a(\zeta) & 1 \\ 0 & 0 \end{pmatrix},
\]

(4.5)

\[
C(\zeta) = A(\zeta) + \pi i B(\zeta) = \begin{pmatrix} \pi i a(\zeta) & \pi i \\ b(\zeta) & -1 \end{pmatrix},
\]

and

(4.6)

\[
D(\zeta) = A(\zeta) - \pi i B(\zeta) = \begin{pmatrix} -\pi i a(\zeta) & -\pi i \\ b(\zeta) & -1 \end{pmatrix}.
\]

In order to insure the existence of the inverses $C^{-1}(\zeta)$ and $D^{-1}(\zeta)$ we suppose that
(4.7) \[ a(\zeta) \neq -b(\zeta). \]

For the present system (4.2)

\[
C^{-1}(\zeta) = -\frac{1}{\pi i[a(\zeta) + b(\zeta)]} \begin{pmatrix} -1 & -\pi i \\ -b(\zeta) & \pi i a(\zeta) \end{pmatrix},
\]

\[
D^{-1}(\zeta) = \frac{1}{\pi i[a(\zeta) + b(\zeta)]} \begin{pmatrix} -1 & \pi i \\ -b(\zeta) & -\pi i a(\zeta) \end{pmatrix};
\]

and

\[
C^{-1}(\zeta) + D^{-1}(\zeta) = \frac{1}{\pi i[a(\zeta) + b(\zeta)]} \begin{pmatrix} 0 & 2\pi i \\ 0 & -2\pi i a(\zeta) \end{pmatrix}.
\]

Hence

\[
[C^{-1}(\zeta) + D^{-1}(\zeta)] f(\zeta) = 0
\]

and

\[
g(z) = C^{-1}(z) f(z) = \frac{f_1(z)}{\pi i[a(z) + b(z)]} \begin{pmatrix} 1 \\ b(z) \end{pmatrix}.
\]

Therefore, from (2.16) the solution formula for (4.2) or (4.1) is

\[
\phi(\zeta) = \frac{[R^+(\zeta) - R^-(\zeta)]}{2\pi i} \left\{ \int_L \frac{[R^+(z)]^{-1} g(z) dz}{z - \zeta} + p(\zeta) \right\}
\]

subject to the conditions on \( g(z) \) and \( p(w) \) noted in Section 2.

The vector

\[
\psi(w) = \begin{pmatrix} \psi_1(w) \\ \psi_2(w) \end{pmatrix}
\]
must satisfy
\[
\psi^+(\zeta) = E(\zeta)\psi^-(\zeta) = C^{-1}(\zeta)D(\zeta)\psi^-(\zeta)
\]

which is the same as

\[
(4.9) \quad [A(\zeta) + \pi\i B(\zeta)]\psi^+(\zeta) = [A(\zeta) - \pi\i B(\zeta)]\psi^-(\zeta).
\]

With the matrices (4.4), equation (4.9) is

\[
\begin{pmatrix}
\pi a(\zeta) & \pi \i \\
b(\zeta) & -1
\end{pmatrix}
\begin{pmatrix}
\psi_1^+(\zeta) \\
\psi_2^+(\zeta)
\end{pmatrix}
= \begin{pmatrix}
-\pi a(\zeta) & -\pi \i \\
b(\zeta) & -1
\end{pmatrix}
\begin{pmatrix}
\psi_1^-(\zeta) \\
\psi_2^-(\zeta)
\end{pmatrix}
\]

which gives

\[
\begin{cases}
\psi_2^+(\zeta) + \psi_2^-(\zeta) = -a(\zeta)[\psi_1^+(\zeta) + \psi_1^-(\zeta)] \\
\psi_2^+(\zeta) - \psi_2^-(\zeta) = b(\zeta)[\psi_1^+(\zeta) - \psi_1^-(\zeta)]
\end{cases}
(4.10)
\]

By adding and subtracting the equations in (4.10) we find

\[
\begin{cases}
2\psi_2^+(\zeta) = [b(\zeta) - a(\zeta)]\psi_1^+(\zeta) - [a(\zeta) + b(\zeta)]\psi_1^-(\zeta) \\
2\psi_2^-(\zeta) = -[a(\zeta) + b(\zeta)]\psi_1^+(\zeta) + [b(\zeta) - a(\zeta)]\psi_1^-(\zeta)
\end{cases}
(4.11)
\]

The equation for

\[
\ell(w) = \frac{\psi_1(w)}{\psi_2(w)}
\]

is

\[
(4.12) \quad 2a(\zeta)b(\zeta)\ell^+(\zeta)\ell^-(\zeta) + [b(\zeta) - a(\zeta)][\ell^+(\zeta) + \ell^-(\zeta)] - 2 = 0.
\]
We also have from (4.10):

\[ a(\zeta) = \frac{-\left[\psi_2^+(\zeta) + \psi_2^-(\zeta)\right]}{\psi_1^+(\zeta) + \psi_1^-(\zeta)} . \]

(4.13)

\[ b(\zeta) = \frac{\psi_2^+(\zeta) - \psi_2^-(\zeta)}{\psi_1^+(\zeta) - \psi_1^-(\zeta)} . \]

(4.14)

\[ a(\zeta)b(\zeta) = \frac{\left[\psi_2^-(\zeta)\right]^2 - \left[\psi_2^+(\zeta)\right]^2}{\left[\psi_1^+(\zeta)\right]^2 - \left[\psi_1^-(\zeta)\right]^2} . \]

(4.15)

\[ b(\zeta) - a(\zeta) = \frac{2\left[\psi_1^+(\zeta)\psi_2^+(\zeta) - \psi_1^-(\zeta)\psi_2^-(\zeta)\right]}{\left[\psi_1^+(\zeta)\right]^2 - \left[\psi_1^-(\zeta)\right]^2} . \]

(4.16)

If \( a(\zeta) \) and \( b(\zeta) \) are arbitrarily assigned we cannot expect to solve (4.10) or (4.12) in closed form. This can be achieved, however, if \( a(\zeta) \) and \( b(\zeta) \) are related in certain special ways. We intend to show that the integral equation (4.1) can be solved in closed form if \( a(\zeta), b(\zeta), \phi(\zeta), f(\zeta) \) satisfy the conditions stipulated in Section 2; if (4.7) holds; and if

\[ a(\zeta)b(\zeta) = s(\zeta)[b(\zeta) - a(\zeta)] + s^2(\zeta) - [(\zeta-\alpha)(\zeta-\beta)]^\kappa t^2(\zeta) \]

\[ s(\zeta) = \frac{s_1(\zeta)}{s_2(\zeta)} ; \quad t(\zeta) = \frac{t_1(\zeta)}{t_2(\zeta)} \]

where \( s_1(\zeta), s_2(\zeta), t_1(\zeta), t_2(\zeta) \) are polynomials and

\[ \kappa = 0 \text{ or } 1. \]
The condition (4.17) relates $a(\zeta)$ to $b(\zeta)$ and it can be expressed as

$$[s(\zeta) - a(\zeta)][s(\zeta) + b(\zeta)] = [(\zeta - \alpha)(\zeta - \beta)]^\kappa t^2(\zeta).$$

If $\kappa = 0$ it can be expressed as

$$(4.19) \quad s_2(\zeta)a(\zeta)b(\zeta) = s_1(\zeta)[b(\zeta) - a(\zeta)] + q(\zeta)[2s_1(\zeta) - s_2(\zeta)q(\zeta)]
$$

where $q(\zeta)$ is a ratio of polynomials. These expressions show that the condition (4.17) covers the following interesting special cases:

1. It covers the cases in which $a(\zeta)$ or $b(\zeta)$ is a ratio of polynomials. For example, suppose $a(\zeta) = s(\zeta)$. Then (4.17) is satisfied when $t = 0$.

2. If $s_2 = 0$ and $s_1 = 1$, (4.19) shows that we have

$$b(\zeta) - a(\zeta) = -2q(\zeta),$$

the case in which the difference $b(\zeta) - a(\zeta)$ is a rational function, and this includes the case

$$a(\zeta) = b(\zeta).$$

3. If $s = 0$, (4.17) shows that we have either

$$a(\zeta)b(\zeta) = -[(\zeta - \alpha)(\zeta - \beta)]t^2(\zeta)$$

or

$$a(\zeta)b(\zeta) = [it(\zeta)]^2.$$
The latter is the case in which the product \( a(\zeta)b(\zeta) \) is the square of a rational function and this includes the case \[ a(\zeta)b(\zeta) = 1. \]

4. If \( q = 0 \) we have
\[ s_2(\zeta)a(\zeta)b(\zeta) = s_1(\zeta)[b(\zeta) - a(\zeta)], \]
the case in which the ratio of the product to the difference of \( a(\zeta) \) and \( b(\zeta) \) is a rational function.

The simplest of the cases enumerated above are those in which either \( a(\zeta) \) or \( b(\zeta) \) is a rational function. When this is the case the solution of (4.10) offers no difficulty. In fact, if \( a(\zeta) \) or \( b(\zeta) \) is equal to the rational function \( s(\zeta) \) then (4.1) can be reduced to a Cauchy singular integral equation of the first kind whose solution is well known. For example if \( a(\zeta) = s(\zeta) \) then (4.1) can be expressed as
\[
\int_{L} \frac{[s(z) + b(z)]\phi_1(z)dz}{z - \zeta} = f_1(\zeta) + \int_{L} \frac{[s(z) - s(\zeta)]\phi_1(z)dz}{z - \zeta},
\]
where the second member of the right-hand side is a rational function of \( \zeta \) involving coefficients \( \lambda_1 \) which depend on \( \phi_1 \). If a solution of the last equation exists it must have the form
\[
\frac{1}{\pi \sqrt{(z-\alpha)(z-\beta)}} \left\{ \int \frac{\sqrt{(\zeta-\alpha)(\zeta-\beta)} f(\zeta) d\zeta}{\zeta-z} + c \right\} \\
[s(z)+b(z)]\phi(z) = \\
\frac{1}{\pi \sqrt{(z-\alpha)(z-\beta)}} \int \frac{\sqrt{(\zeta-\alpha)(\zeta-\beta)} [s(t)-s(\zeta)] \phi_1(t) dt d\zeta}{t-\zeta}
\]

from which the values of \( \lambda_1 \) can be deduced. For the above reasons we will forego any further discussion of Case 1. In other words we will hereinafter suppose that \( t \neq 0 \).

We proceed to derive the fundamental matrix functions for the remaining cases. After that we will present some specific examples of the solution formula (4.8).

If we use (4.15) and (4.16) we find that the condition (4.17) requires

\[
[\psi^+_2(\zeta) + s(\zeta)\psi^+_1(\zeta)]^2 - [((\zeta-\alpha)(\zeta-\beta))]^\kappa [t(\zeta)\psi^+_1(\zeta)]^2
\]

(4.20)

\[
= [\psi^-_2(\zeta) + s(\zeta)\psi^-_1(\zeta)]^2 - [((\zeta-\alpha)(\zeta-\beta))]^\kappa [t(\zeta)\psi^-_1(\zeta)]^2.
\]

The general solution of this equation is

(4.21) \[ [\psi_2(w) + s(w)\psi_1(w)]^2 - [(w-\alpha)(w-\beta)]^\kappa [t(w)\psi_1(w)]^2 = \mu(w) \]

where 
\[ \mu^+(\zeta) = \mu^-_*(\zeta) = \mu(\zeta) . \]

Consequently,

(4.22) \[ [\psi^+_2(\zeta) + s(\zeta)\psi^+_1(\zeta)]^2 - [((\zeta-\alpha)(\zeta-\beta))]^\kappa [t(\zeta)\psi^+_1(\zeta)]^2 = \mu(\zeta) . \]
With (4.22) we have, from (4.11),

\[ 2\psi_2^+(\zeta) = [b(\zeta) - a(\zeta)]\psi_1^+(\zeta) - [a(\zeta) - b(\zeta)]\psi_1^-(\zeta). \]  

The elimination of \( \psi_2^+(\zeta) \) from (4.22) and (4.23) gives

\[
\left\{ \begin{array}{l}
[b(\zeta) - a(\zeta)]^2 + 4s(\zeta)[b(\zeta) - a(\zeta)] + 4s^2(\zeta) \\
\quad - 4[(\zeta-\alpha)(\zeta-\beta)]^2t^2(\zeta)
\end{array} \right\} [\psi_1^+(\zeta)]^2 = 4\mu(\zeta)
\]

\[ -2[a(\zeta) + b(\zeta)][b(\zeta) - a(\zeta) + 2s(\zeta)]\psi_1^+(\zeta)\psi_1^-(\zeta) + [a(\zeta) + b(\zeta)]^2[\psi_1^-(\zeta)]^2 \]

and if we use (4.17), namely,

\[ s(\zeta)[b(\zeta) - a(\zeta)] + s^2(\zeta) - [(\zeta-\alpha)(\zeta-\beta)]^2t^2(\zeta) = a(\zeta)b(\zeta) \]

we are led to

\[ [a(\zeta) - s(\zeta)][\psi_1^+(\zeta) + \psi_1^-(\zeta)]^2 + [b(\zeta) + s(\zeta)][\psi_1^+(\zeta) - \psi_1^-(\zeta)]^2 \]

\[ = \frac{4\mu(\zeta)}{a(\zeta) + b(\zeta)}. \]

We need solutions \( \psi_1(w) \) of this nonlinear barrier equation which behave like polynomials at infinity while they are analytic in \( \Lambda \) which is the finite complex plane slit along the path \( L \). Once such solutions have been found they determine \( \psi_2(w) \) through equation
(4.21). Then $\psi_1(w)$ and $\psi_2(w)$ can presumably be used to construct the fundamental matrix function $R(w)$.

First let us examine the consequences of taking $\mu(w) = 0$. If $
mu(w) = 0$ the barrier equation (4.25) becomes

$$(4.26) \quad \sqrt{s(\zeta) - a(\zeta)} [\psi_1^+(\zeta) + \psi_1^-(\zeta)] = \pm \sqrt{s(\zeta) + b(\zeta)} [\psi_1^+(\zeta) - \psi_1^-(\zeta)]$$

which is the linear equation

$$(4.27) \quad \left[ \sqrt{s(\zeta) - a(\zeta)} \pm \sqrt{s(\zeta) + b(\zeta)} \right] \psi_1^+(\zeta) = - \left[ \sqrt{s(\zeta) - a(\zeta)} \pm \sqrt{s(\zeta) + b(\zeta)} \right] \psi_1^-(\zeta)$$

As we have noted, we are now assuming that $\tau(\zeta) \neq 0$, so $a(\zeta)$ is not $s(\zeta)$ nor is $b(\zeta)$ equal to $-s(\zeta)$. The general solution of equation (4.27) is

$$(4.28) \quad \psi_1(w) = \sigma(w) \exp \left[ \pm \frac{1}{2\pi i} \int_{L} \frac{\ln N(z)dz}{z-w} \right]$$

where

$$(4.29) \quad N(z) = \frac{\sqrt{s(z) + b(z)} + \sqrt{s(z) - a(z)}}{\sqrt{s(z) + b(z)} - \sqrt{s(z) - a(z)}}$$

and

$$\sigma^+(\zeta) = \sigma^-(\zeta) = \sigma(\zeta).$$

With $\mu = 0$ equation (4.21) yields

$$\psi_2(w) = \left\{ -s(w) \pm t(w)[(w-a)(w-\beta)]^{\kappa/2} \right\} \psi_1(w)$$

and so from (4.28)
\[ (4.30) \quad \psi_2(w) = \sigma(w) \left\{ -s(w) \pm t(w) \left[ (w-\alpha)(w-\beta) \right]^{\kappa/2} \right\} \]

\[ \cdot \exp \left[ \pm \frac{1}{2\pi i} \int_L \frac{\ln N(z)dz}{z-w} \right]. \]

It can be verified that (4.28) and (4.30) satisfy the system (4.10), that is,

\[ \psi_2^+(\zeta) + \psi_2^-(\zeta) = -a(\zeta)[\psi_1^+(\zeta) + \psi_1^-(\zeta)] \]

\[ \psi_2^+(\zeta) - \psi_2^-(\zeta) = b(\zeta)[\psi_1^+(\zeta) - \psi_1^-(\zeta)] \]

subject to

\[ a(\zeta)b(\zeta) = s(\zeta)[b(\zeta) - a(\zeta)] + s^2(\zeta) - [(\zeta-\alpha)(\zeta-\beta)]^{\kappa} t^2(\zeta), \]

where \( t \neq 0, \) provided \( \kappa = 1. \) If \( \kappa = 0 \) we have a situation which can be covered by assuming \( \mu(w) \neq 0, \) as we will see later.

The value of the multi-valued function which appears in

\[ (4.31) \quad h(w) = \exp \left[ \pm \frac{1}{2\pi i} \int_L \frac{\ln N(z)dz}{z-w} \right] \]

is assumed to be such that

\[ \ln N(z) = \ln \left[ \frac{\sqrt{s(z)+b(z)} + \sqrt{s(z)-a(z)}}{\sqrt{s(z)+b(z)} - \sqrt{s(z)-a(z)}} \right] \]

\[ = \ln \left\{ \frac{t(z)(z-\alpha)(z-\beta) + [s(z)-a(z)]}{t(z)(z-\alpha)(z-\beta) - [s(z)-a(z)]} \right\} \]

varies continuously from its initial value \( \ln N(\alpha) \) to its terminal
value \ln N(\beta) as z moves from \alpha along L to \beta. The initial value

\[ \ln N(\alpha) = \ln |N(\alpha)| + i \arg N(\alpha) = (\ell_1 + i\theta_1)2\pi \]

will be taken so that \(0 < \theta_1 < 1\). In fact, since \(t\) can be replaced by \(-t\) without disturbing what we have so far, we can take

\[ 0 < \theta_1 < \frac{1}{2} \]

as we will. For the terminal value

\[ \ln N(\beta) = (\ell_2 + i\theta_2)2\pi \]

we cannot confine \(\theta_2\) to the same range as \(\theta_1\) because the point representing \(N(z)\) may wind around the origin as \(z\) traverses \(L\).

The behavior of \(h(w)\) in the neighborhood of \(\alpha\) is given by

\[
\begin{align*}
    h(w) &= \exp \left\{ \frac{\ln N(\alpha)}{2\pi i} \int_{L} \frac{dz}{z-w} + \frac{1}{2\pi i} \int_{L} \frac{[\ln N(z) - \ln N(\alpha)]dz}{z-w} \right\} \\
    &= h_1(w) \exp \left\{ -\frac{(\ell_1 + i\theta_1)2\pi \cdot \ln (\alpha-w)}{2\pi i} \right\} \\
    &= \frac{h_1(w)}{(\alpha-w)^{\theta_1-\ell_1}}
\end{align*}
\]

where \(h_1(w)\) is defined, not zero, and bounded at \(w = \alpha\). The behavior of \(h(w)\) in the neighborhood of \(\beta\) is given by

\[ h(w) = (\beta-w)^{\theta_2-\ell_2} h_2(w) \]
where \( h_2(w) \) is defined, not zero, and bounded at \( w = \beta \). Thus we see that the boundedness of \( h(w) \) and \( 1/h(w) \) at the end points of \( L \) depends on \( \theta_1 \) and \( \theta_2 \). Note that if \( m \) is an integer such that

\[
(4.32) \quad -\frac{1}{2} \leq m - \theta_2 \leq \frac{1}{2}
\]

then

\[
\frac{(w-\beta)^m \sqrt{(w-\alpha)(w-\beta)}}{h(w)}
\]

and

\[
\frac{h(w) \sqrt{(w-\alpha)(w-\beta)}}{(w-\beta)^m}
\]

are bounded at the end points of \( L \).

We turn now to the formation of \( R(w) \) from \( \psi_1(w) \) and \( \psi_2(w) \) when \( a(\zeta) \) and \( \beta(\zeta) \) are so related that

\[
a(\zeta)b(\zeta) = s(\zeta)[b(\zeta) - a(\zeta)] + s^2(\zeta) - (\zeta-\alpha)(\zeta-\beta)t^2(\zeta)
\]

where \( t(\zeta) \neq 0 \). The functions \( \psi_1(w) \) and \( \psi_2(w) \) are given by (4.28), (4.30) and (4.31). Let us take

\[
R(w) = \begin{pmatrix}
\sigma_1(w)h(w) & \frac{\sigma_2(w)}{h(w)} \\
\sigma_1(w)[-s(w)+t(w)\sqrt{(w-\alpha)(w-\beta)}]h(w) & \frac{\sigma_2(w)[-s(w)-t(w)\sqrt{(w-\alpha)(w-\beta)}]}{h(w)}
\end{pmatrix}
\]

where \( \sigma_1(w) \) and \( \sigma_2(w) \) must satisfy
\[ \sigma_1^+(\zeta) = \sigma_1^-(\zeta) = \sigma_1(\zeta) ; \quad \sigma_2^+(\zeta) = \sigma_2^-(\zeta) = \sigma_2(\zeta) \]

but are otherwise at our disposal. The inverse of this \( R(w) \) is

\[
R^{-1}(w) = -\frac{1}{\pi i} \begin{pmatrix}
\frac{-s - t \sqrt{(w-\alpha)(w-\beta)}}{\sigma_1 \chi \sqrt{(w-\alpha)(w-\beta)}} & -\frac{1}{\sigma_1 \chi \sqrt{(w-\alpha)(w-\beta)}} \\
\frac{-[s + t \sqrt{(w-\alpha)(w-\beta)}]h(w)}{\sigma_2 \chi \sqrt{(w-\alpha)(w-\beta)}} & \frac{h(w)}{\sigma_2 \chi \sqrt{(w-\alpha)(w-\beta)}}
\end{pmatrix}
\]

We require \( R^{-1}(w) \) to be analytic in \( \Lambda \), bounded on \( L \), and of finite degree at infinity. These requirements are satisfied if we take

\[
\frac{1}{\sigma_1(w)} = s_2(w)t_1(w)(w-\alpha)(w-\beta)(w-\beta)^m
\]

\[
\frac{1}{\sigma_2(w)} = s_2(w)t_1(w) \frac{(w-\alpha)(w-\beta)}{(w-\beta)^m}.
\]

Then

(4.33) \( R^{-1}(w) \)

\[
R^{-1}(w) = -\frac{1}{\pi i} \begin{pmatrix}
\frac{[-s_1 t_2 - t_1 \sqrt{(w-\alpha)(w-\beta)}](w-\beta)^m \sqrt{(w-\alpha)(w-\beta)}}{h} & -s_2 t_2 \frac{(w-\beta)^m \sqrt{(w-\alpha)(w-\beta)}}{h(w)} \\
\frac{[-s_1 t_2 + t_1 \sqrt{(w-\alpha)(w-\beta)}]h(w)/(w-\alpha)(w-\beta)}{(w-\beta)^m} & s_2 t_2 h \frac{\sqrt{(w-\alpha)(w-\beta)}}{(w-\beta)^m}
\end{pmatrix}
\]

and
(4.34) \[ R(w) = \begin{pmatrix} \frac{h(w)}{s_2 t_1(w-\alpha)(w-\beta)^{m+1}} & \frac{(w-\beta)^{m-1}}{s_2 t_1(w-\alpha)h(w)} \\ \frac{[-s+t\sqrt{(w-\alpha)(w-\beta)}]h(w)}{s_2 t_1(w-\alpha)(w-\beta)^{m+1}} & \frac{(w-\beta)^{m-1}[s-t\sqrt{(w-\alpha)(w-\beta)}]}{s_2 t_1(w-\alpha)h(w)} \end{pmatrix} \]

where \( h(w) \) is defined by (4.31).

The above analysis demonstrates that if there exists a solution of

\[ (4.35) \quad a(\zeta) \int \frac{\phi_1(z)dz}{L(z-\zeta)} + \int \frac{b(z)\phi_1(z)dz}{L(z-\zeta)} = f_1(\zeta) \]

subject to

\[ (4.36) \quad a(\zeta)b(\zeta) = s(\zeta)[b(\zeta) - a(\zeta)] + s^2(\zeta) - (\zeta - \alpha)(\zeta - \beta)t^2(\zeta), \quad t(\zeta) \neq 0, \]

then it can be expressed in the form (4.8), namely,

\[ (4.37) \quad \phi(\zeta) = \left[ \frac{[R^+(\zeta) - R^-(\zeta)]}{2\pi i} \right] \left\{ \int \frac{[R^+(z)]^{-1}g(z)dz}{z - \zeta} + p(\zeta) \right\} \]

where \( R^{-1}(w) \) is the matrix function defined by (4.33) and

\[ \phi(\zeta) = \begin{pmatrix} \phi_1(\zeta) \\ b(\zeta)\phi_1(\zeta) \end{pmatrix}, \quad g(z) = \frac{f_1(z)}{\pi i[a(z) + b(z)]}, \quad \begin{pmatrix} 1 \\ b(z) \end{pmatrix} \]

while \( p(\zeta) \) is the vector polynomial.
\[ p(\zeta) = \begin{pmatrix} p_1(\zeta) \\ p_2(\zeta) \end{pmatrix} \]

The functions \( g(z) \) and \( p(\zeta) \) must also satisfy the conditions enunciated at the end of Section 2.

As an illustration let us apply the foregoing to the solution of the equation

\[
\sqrt{1-\zeta^2} \int_{-1}^{1} \frac{\phi_1(z)dz}{z-\zeta} + \lambda \int_{-1}^{1} \frac{\sqrt{1-z^2}}{z-\zeta} = f_1(\zeta)
\]

where \( \lambda \) is real and \( \lambda > 0 \).

Here we have

\[
a(\zeta) = \sqrt{1-\zeta^2} ; \quad b(\zeta) = \lambda^2 \sqrt{1-\zeta^2} .
\]

These functions satisfy

\[
ab = \frac{s_1}{s_2} (b-a) + \frac{s_1}{s_2} - (\zeta+1)(\zeta-1) \frac{t_1^2}{t_2}
\]

is we take

\[
s_1 = 0 ; \quad s_2 = 1 ; \quad t_1 = \lambda ; \quad t_2 = 1 .
\]

Hence

\[
\ln N(z) = \ln \left( \frac{\sqrt{s+b} + \sqrt{s-a}}{\sqrt{s+b} - \sqrt{s-a}} \right)
\]

\[
= \ln \left( \frac{\lambda+1}{\lambda-1} \right) = i\theta 2\pi
\]

30
from which $\theta$ is constant and

$$0 < \theta < \frac{1}{2}.$$ 

Therefore

$$h(w) = \exp \left[ \frac{1}{2\pi i} \int_{-1}^{1} \frac{\ln N(z)dz}{z-w} \right] = \left( \frac{w-1}{w+1} \right)^\theta$$

and we see that (4.32) is satisfied with $m = 0$. From (4.33) the matrix function $R^{-1}(w)$ now becomes

(4.39) $$R^{-1}(w) = -\frac{1}{2} \begin{pmatrix} -\lambda(w+1)^{1+\theta}(w-1)^{1-\theta} & (w+1)^{1+\theta}(w-1)^{1-\theta} \\ -\lambda(w+1)^{1-\theta}(w-1)^{1+\theta} & (w+1)^{1-\theta}(w-1)^{1+\theta} \end{pmatrix}$$

while $R(w)$ becomes

(4.40) $$R(w) = \begin{pmatrix} \frac{1}{\lambda(w+1)^{1+\theta}(w-1)^{1-\theta}} & \frac{1}{\lambda(w+1)^{1-\theta}(w-1)^{1+\theta}} \\ \frac{1}{(w+1)^{1+\theta}(w-1)^{1-\theta}} & \frac{1}{(w+1)^{1-\theta}(w-1)^{1+\theta}} \end{pmatrix}.$$ 

From an examination of the behavior of

$$R^{-1}(w) = \int_{L} \frac{\phi(z)dz}{z-w}$$

as $w \to \infty$ we find that $p(w)$ must be
The substitution of (4.39), (4.40) and (4.41) in (4.37) shows, after some simplification, that the solution $\phi_1(\zeta)$ of (4.38) can be expressed in the form

$$
\phi_1(\zeta) = \frac{1}{2\pi^2(1+\lambda^2)} \left[ \frac{1}{(1+\zeta)^\theta (1-\zeta)^{1-\theta}} \int_{-1}^{1} \frac{(1+z)^{-\frac{1}{2}}(1-z)^{\frac{1}{2}}}{z^{1-\theta}} f_1(z) \, dz + \frac{1}{(1+\zeta)^{1-\theta} (1-\zeta)^{1-\theta}} \int_{-1}^{1} \frac{(1+z)^{\frac{1}{2}}}{z^{1-\theta}} (1-z)^{-\theta} f_1(z) \, dz \right] 

+ c \left[ \frac{1}{(1+\zeta)^\theta (1-\zeta)^{1-\theta}} + \frac{1}{(1+\zeta)^{1-\theta} (1-\zeta)^\theta} \right].
$$

This formula contains only one arbitrary constant whereas there are apparently three in (4.41). This is so because special values of $c_2$ and $c_3$ are needed in order to insure that $\phi_1(\zeta)$ has admissible singularities at the end points -1, and 1.
5. The kernel \([a(\zeta) + b(z)]/(z - \zeta)\) subject to
\([s(\zeta) - a(\zeta)][s(\zeta) + b(\zeta)] = t^2(\zeta)\)

We have shown how

\[(5.1) \quad a(\zeta) \int_L \frac{\phi_1(z)dz}{z - \zeta} + \int_L \frac{b(z)\phi_1(z)dz}{z - \zeta} = f_1(\zeta)\]

can be solved when \(a(\zeta)\) and \(b(\zeta)\) satisfy \((4.17)\) with \(\kappa = 1\). Let us continue by investigating the solution of \((5.1)\) when \(\kappa = 0\), that is, when

\[(5.2) \quad a(\zeta)b(\zeta) = s(\zeta)[b(\zeta) - a(\zeta)] + s^2(\zeta) - t^2(\zeta)\]

which is the same as

\[(5.3) \quad [s(\zeta) - a(\zeta)][s(\zeta) + b(\zeta)] = t^2(\zeta)\]

where

\[s(\zeta) = \frac{s_1(\zeta)}{s_2(\zeta)}; \quad t(\zeta) = \frac{t_1(\zeta)}{t_2(\zeta)}\]

are ratios of polynomials.

Equations \((4.25)\) and \((4.21)\) show that the equations for \(\psi_1(w)\) and \(\psi_2(w)\) when \(\kappa = 0\) are

\[(5.4) \quad [a(\zeta) - s(\zeta)][\psi^+_1(\zeta) + \psi^-_1(\zeta)]^2 + [b(\zeta) + s(\zeta)][\psi^+_1(\zeta) - \psi^-_1(\zeta)]^2 = \frac{\mu_\mu(\zeta)}{a(\zeta) + b(\zeta)}\]

and
\( (5.5) \quad \psi_2(w) = -s(w)\psi_1(w) \pm \sqrt{t^2(w)\psi_1^2(w) + \mu(w)} \)

where

\[ \mu^+(\zeta) = \mu^-(\zeta) = \mu(\zeta). \]

If we take

\[ \mu(w) = -\sigma^2(w)t^2(w) \]

where

\[ \sigma^+(\zeta) = \sigma^-(\zeta) = \sigma(\zeta), \]

then solutions of (5.4) and (5.5), if they exist, must have the form

\( (5.6) \quad \psi_1(w) = \sigma(w)\bar{\psi}_1(w) \)

and

\( (5.7) \quad \psi_2(w) = \sigma(w)\left[ -s(w)\bar{\psi}_1(w) \pm t(w)\sqrt{\frac{1}{\psi_1^2(w)} - 1} \right] \).

The requirement that \( \psi_2(w) \) be analytic in \( \Lambda \), and the expression (5.7) suggest looking for a solution of

\( (5.8) \quad [a(\zeta) - s(\zeta)][\psi_1^+(\zeta) + \psi_1^-(\zeta)]^2 + [b(\zeta) + s(\zeta)][\psi_1^+(\zeta) - \psi_1^-(\zeta)]^2 = \frac{-4\sigma^2(\zeta)t^2(\zeta)}{a(\zeta) + b(\zeta)} \)

in the form

\( (5.9) \quad \psi_1(w) = \sigma(w) \cosh u(w) \)

where

\( (5.10) \quad u(w) = \sqrt{(w-\alpha)(w-\beta)} \int_{L} \frac{v(z)dz}{\sqrt{(z-\alpha)(z-\beta)(z-w)}} \)

34
and \( v(z) \) is to be determined. Since

\[
\begin{align*}
    u^+(\zeta) &= v(\zeta) + \frac{\sqrt{(\zeta-\alpha)(\zeta-\beta)}}{\pi L} \int_{L}^{\infty} \frac{v(z)dz}{\sqrt{(z-\alpha)(z-\beta)(z-\zeta)}} \\
    &= v(\zeta) + u(\zeta)
\end{align*}
\]

and

\[
    u^-(\zeta) = v(\zeta) - u(\zeta)
\]

we find from (5.9) that

\[
(5.11) \quad [\psi^+(\zeta) + \psi^-_1(\zeta)]^2 = 4r^2(\zeta) \cosh^2 v(\zeta) \cosh^2 u(\zeta)
\]

and

\[
(5.12) \quad [\psi^+(\zeta) - \psi^-_1(\zeta)]^2 = 4r^2(\zeta) \sinh^2 v(\zeta) \sinh^2 u(\zeta).
\]

With (5.11) and (5.12) equation (5.8) becomes

\[
(5.13) \quad [a(\zeta) - s(\zeta)] \cosh^2 v(\zeta) \cosh^2 u(\zeta) = \frac{-r^2(\zeta)}{a(\zeta) + b(\zeta)} \\
+ [b(\zeta) + s(\zeta)] \sinh^2 v(\zeta) \sinh^2 u(\zeta)
\]

which can be reduced to

\[
[cosh 2v + \frac{a - b - 2s}{a + b}][cosh 2u + \frac{a - b - 2s}{a + b}] = 0.
\]

This shows that (5.13) is satisfied if

\[
cosh 2v = \frac{2s + b - a}{a + b},
\]

that is, if
\begin{equation}
(5.14) \quad \nu(\zeta) = \pm \frac{1}{2} \ln \left[ \frac{s(\zeta) + b(\zeta) + \sqrt{s(\zeta) - a(\zeta)}}{s(\zeta) + b(\zeta) - \sqrt{s(\zeta) - a(\zeta)}} \right]
= \pm \frac{1}{2} \ln N(\zeta) .
\end{equation}

We have now found that if

\begin{equation}
(5.15) \quad u(w) = \sqrt{(w-a)(w-b)} \int_{L} \frac{\ln N(z) dz}{(z-a)(z-b)(z-w)}
\end{equation}

then

\begin{equation}
(5.16) \quad \psi_1(w) = \sigma(w) \cosh u(w)
\end{equation}

satisfies (5.8); and \( \psi_2(w) \) from (5.7) is

\begin{equation}
(5.17) \quad \psi_2(w) = \sigma(w)[-s(w) \cosh u(w) \pm t(w) \sinh u(w)] .
\end{equation}

As in Section 4 we assume here also that \( \ln N(z) \) varies continuously as \( z \) moves along \( L \) from \( \alpha \) to \( \beta \). There is no problem with respect to the singular behavior of \( \psi_1 \) and \( \psi_2 \) at the end points of \( L \) because \( u(w) \) is bounded at these points. It is now easy to check that (5.16) and (5.17) satisfy (4.10).

If we try to construct \( R(w) \) from the solutions we have just found we are led to

\[
R(w) = \begin{pmatrix}
\sigma_1 \cosh u & \sigma_2 \cosh u \\
\sigma_1[-s \cosh u + t \sinh u] & \sigma_2[-s \cosh u - t \sinh u]
\end{pmatrix} .
\]

The inverse of this is
\[
R^{-1}(w) = \begin{pmatrix}
\frac{s(w) \cosh u(w) + t(w) \sinh u(w)}{2\sigma_1(w) t(w) \sinh u(w) \cosh u(w)} & \frac{1}{2\sigma_1(w) t(w) \sinh u(w)} \\
\frac{s(w) \cosh u(w) - 2(w) \sinh u(w)}{2\sigma_2(w) t(w) \sinh u(w) \cosh u(w)} & -\frac{1}{2\sigma_2(w) t(w) \sinh u(w)}
\end{pmatrix}.
\]

However, a difficulty with this is that we cannot tell where the zeros of \(\sinh u(w)\) and \(\cosh u(w)\) are located.

Another kind of solution of the equations (5.4) and (5.5) can be found if we set

\[
\mu(w) = -\frac{\tilde{\sigma}^2(w) t^2(w)}{(w-\alpha)(w-\beta)}
\]

\[
\tilde{\sigma}^+(\xi) = \tilde{\sigma}^-(\xi) = \tilde{\sigma}(\xi)
\]

so that the equations for \(\psi_1(w)\) and \(\psi_2(w)\) become

\[
(5.18) \quad [a(\xi) - s(\xi)][\psi_1^+(\xi) + \psi_1^-(\xi)]^2 + [b(\xi) + s(\xi)][\psi_1^+(\xi) - \psi_1^-(\xi)]^2
\]

\[
= -\frac{4\tilde{\sigma}^2(\xi) t^2(\xi)}{[a(\xi) + b(\xi)](\xi-\alpha)(\xi-\beta)}
\]

and

\[
(5.19) \quad \psi_2(w) = -s(w)\psi_1(w) \pm \frac{t(w)\sqrt{(w-\alpha)(w-\beta)\psi_1^2(w) - \tilde{\sigma}^2(w)}}{\sqrt{(w-\alpha)(w-\beta)}}.
\]

If we let

\[
\psi_1(w) = \frac{\tilde{\sigma}(w)\tilde{\psi}_1(w)}{\sqrt{(w-\alpha)(w-\beta)}} ; \quad \psi_2(w) = \frac{\tilde{\sigma}(w)\tilde{\psi}_2(w)}{\sqrt{(w-\alpha)(w-\beta)}}
\]

we have the following equations for \(\tilde{\psi}_1(w)\) and \(\tilde{\psi}_2(w)\):
(5.20) \[ [a(\zeta) - s(\zeta)][\psi_1^+(\zeta) - \psi_1^-(\zeta)]^2 + [b(\zeta) + s(\zeta)][\psi_1^+(\zeta) + \psi_1^-(\zeta)]^2 \]
\[ = - \frac{4t^2(\zeta)}{a(\zeta) + b(\zeta)}, \]

(5.21) \[ \tilde{\psi}_2(w) = -s(w)\tilde{\psi}_1(w) \pm t(w) \sqrt{\tilde{\psi}_1^2(w) - 1}. \]

It can be shown that if

\[ u(w) = \frac{\sqrt{(w-\alpha)(w-\beta)}}{2\pi i} \int_L \frac{\ln N(z)dz}{\sqrt{(z-\alpha)(z-\beta)(z-w)}} \]

where

\[ N(z) = \left[ \frac{\sqrt{s(z) + b(z)}}{\sqrt{s(z) + b(z)} - \sqrt{s(z) - a(z)}} \right] \]

then

\[ \tilde{\psi}_1(w) = i \sinh u(w) \]

satisfies (5.20). Hence we find

(5.23) \[ \psi_1(w) = \frac{\sigma(w) \sinh u(w)}{\sqrt{(w-\alpha)(w-\beta)}} \]

(5.24) \[ \psi_2(w) = \frac{\sigma(w)[-s(w) \sinh u(w) \pm t(w) \cosh u(w)]}{\sqrt{(w-\alpha)(w-\beta)}} \].

These functions also satisfy (4.10).

The fundamental matrix function \( R(w) \) derived from (5.16), (5.17), (5.23) and (5.24) is
(5.25) \( R(w) = \begin{pmatrix} 
\sigma_1(w) \cosh u(w) \\
\sigma_1(w) [\frac{-s(w) \cosh u(w)}{\sigma_1(w)t(w)}] \\
\sigma_1(w) [\frac{t(w) \sinh u(w)}{\sigma_1(w)t(w)}] 
\end{pmatrix} \frac{\sigma_2(w) \sinh u(w)}{\sqrt{(w-\alpha)(w-\beta)}} \)

It's inverse is

5.26) \( R^{-1}(w) = \begin{pmatrix} 
\frac{-s(w) \sinh u(w)}{\sigma_1(w)t(w)} \\
\frac{+t(w) \cosh u(w)}{\sigma_1(w)t(w)} 
\end{pmatrix} \frac{\sigma_2(w) \sinh u(w)}{\sqrt{(w-\alpha)(w-\beta)}} \frac{s(w) \cosh u(w)}{\sigma_2(w)t(w)} \frac{+t(w) \cosh u(w)}{\sigma_2(w)t(w)} \)

In order to have the inverse bounded along \( L \), analytic in \( \Lambda \), and of least degree at infinity we can take

\( \sigma_1(w) = \sigma_2(w) = \frac{1}{s_2(w)t_1(w)} \)

provided neither \( s_2 \) nor \( t_1 \) is identically zero. Then

5.27) \( R(w) = \frac{1}{s_2(w)t_1(w)} \begin{pmatrix} 
cosh u(w) \\
\frac{-s(w) \cosh u(w)}{\sqrt{(w-\alpha)(w-\beta)}} \\
\frac{+t(w) \cosh u(w)}{\sqrt{(w-\alpha)(w-\beta)}} 
\end{pmatrix} \frac{\frac{-s(w) \sinh u(w)}{\sqrt{(w-\alpha)(w-\beta)}}}{\frac{+t(w) \cosh u(w)}{\sqrt{(w-\alpha)(w-\beta)}}} \frac{+t(w) \cosh u(w)}{\sigma_2(w)t(w)} \)
and the inverse is

(5.28) \( R^{-1}(w) \)

\[
R^{-1}(w) = s_2(w)t_2(w) \begin{bmatrix}
-s(w) \sinh u(w) \\
-t(w) \cosh u(w)
\end{bmatrix} - \sinh u(w)
\begin{bmatrix}
+s(w) \cosh u(w) \\
-t(w) \sinh u(w)
\end{bmatrix}
\frac{1}{(w-a)(w-p)} \begin{bmatrix}
+s(w) \cosh u(w) \\
-t(w) \sinh u(w)
\end{bmatrix} \cosh u(w)
\]

The results (5.27) and (5.28) are not valid for the case 2 in which

\[
a(\zeta) - b(\zeta) = 2q(\zeta) = \frac{2q_1(\zeta)}{q_2(\zeta)}
\]

where \( q(\zeta) \) is a rational function, because the realization of this case only appears, in general, if we substitute

(5.29) \( s = \frac{S_1}{s_2} = \frac{1}{\epsilon} \); \( t = \frac{1}{\epsilon} - q \)

in

\[
a(\zeta)b(\zeta) = s(\zeta)[b(\zeta) - a(\zeta)] + s^2(\zeta) - t^2(\zeta)
\]

and then let \( \epsilon \to 0 \). However, the function \( R(w) \) for the case 2 can be found if we use (5.29) in (5.25) and adjust \( \sigma_1(w) \), \( \sigma_2(w) \) before we let \( \epsilon \to 0 \). If we set

\[
\sigma_1(w) = \sigma_2(w) = \frac{\sqrt{2\epsilon}}{q_2(w)}
\]

and then pass to the limit it turns out that
(5.30) \[ R(w) = \frac{1}{q_2(w)} \begin{pmatrix} \gamma(w) \\ \frac{1}{\gamma(w)} \end{pmatrix} \begin{pmatrix} \frac{\gamma(w)}{\sqrt{(w-\alpha)(w-\beta)}} \\ \frac{1}{\gamma(w)} - q(w) \gamma(w) \end{pmatrix} \]

and

(5.31) \[ R^{-1}(w) = \frac{q_2(q)}{2} \begin{pmatrix} \frac{1}{\gamma(w)} - q(q) \gamma(w) \\ \frac{-\gamma(w)}{\sqrt{(w-\alpha)(w-\beta)}} \end{pmatrix} \begin{pmatrix} \frac{1}{\gamma(w)} + q(w) \gamma(w) \\ \gamma(w) \end{pmatrix} \]

where

(5.32) \[ \gamma(w) = \exp \left[ -\frac{\sqrt{(w-\alpha)(w-\beta)}}{2\pi} \int_L \frac{\ln [a(z) - q(z)]dz}{\sqrt{z-\alpha)(z-\beta)(z-w)}} \right]. \]

Specific formulas for the solution of (5.1) can now be obtained by substituting (5.27) or (5.30) in (4.8). As an illustration consider the case \( a(\zeta) = b(\zeta) \), that is, the equation

(5.33) \[ a(\zeta) \int_L \frac{\phi_1(z)dz}{z-\zeta} + \int_L a(z)\phi_1(z)dz \frac{1}{z-\zeta} = f_1(\zeta). \]

The appropriate function \( R(w) \) is (5.30) with \( q_1 = 0; q_2 = 1 \), namely

(5.34) \[ R(w) = \begin{pmatrix} \gamma(w) \\ \frac{1}{\gamma(w)} \end{pmatrix} \begin{pmatrix} \frac{\gamma(w)}{\sqrt{(w-\alpha)(w-\beta)}} \\ \frac{1}{\gamma(w)/\sqrt{(w-\alpha)(w-\beta)}} \end{pmatrix} \]
where \( \gamma(w) \) is now defined by

\[
(5.35) \quad \gamma(w) = \exp \left[ -\frac{1}{2\pi i} \int_L \frac{\ln a(z)dz}{\sqrt{(w-\alpha)(w-\beta)(z-w)}} \right]
\]

which shows that (5.34) is analytic in the slit plane \( \Lambda \). The inverse of (5.34) is

\[
(5.36) \quad R^{-1}(w) = \frac{\sqrt{(w-\alpha)(w-\beta)}}{2i} \begin{pmatrix} 1 & -\gamma(w) \\ \gamma(w) & \gamma(w) \end{pmatrix}
\]

This is bounded on \( L \), analytic in \( \Lambda \), and of degree one at infinity. It follows that the vector function \( p(\zeta) \) in (4.8) is

\[
p(\zeta) = \begin{pmatrix} 0 \\ k \end{pmatrix}
\]

where \( k \) is an arbitrary constant. The substitution of (5.34) and (5.36) in

\[
\phi(\zeta) = \frac{[R^+(\zeta)-R^-(\zeta)]}{2\pi i} \left[ \int_L \frac{[R^+(z)]^{-1}g(z)dz}{z-\zeta} \right] + p(\zeta)
\]

where

\[
g(z) = \frac{f_1(z)}{2\pi i a(z)} \begin{pmatrix} 1 \\ a(z) \end{pmatrix}
\]

yields

42
for the solution of (5.33).

6. A System of Cauchy Singular Integral Equations

Defined along a Closed Curve.

Let $\Gamma$ be a simple, smooth, closed path. Let the positive side of $\Gamma$ be contained in the bounded domain $\Lambda^+$ whose periphery is $\Gamma$. The complex plane minus $\Gamma + \Lambda^+$ will be denoted by $\Lambda^-$, and this domain contains the point at infinity. If the components of

(6.1) $A(\zeta)\phi(\zeta) + B(\zeta) \int_{\Gamma} \frac{\phi(z)dz}{z-\zeta} = f(\zeta)$

are Hölder continuous on $\Gamma$, then (6.1) can be studied by using the method of Section 2 which shows that if a solution of (6.1) exists then it must have the form

(6.2) $\phi(\zeta) = \frac{1}{2} [C^{-1}(\zeta) + D^{-1}(\zeta)] f(\zeta)$

$+ \frac{[R^+(\zeta) - R^-(\zeta)]}{2\pi i} \int_{\Gamma} \frac{[R^+(z)]^{-1}c^{-1}(z)f(z)dz}{z-\zeta} + p(\zeta)$.
Here and in what follows the matrices may be $m \times m$, $m \geq 1$. The form (6.2) is closed if we can express the fundamental matrix function $R(w)$ in terms of a finite number of quadratures. (In general, it is not possible to do this.) The resolvent matrix function must satisfy

$$R^+(\zeta) = E(\zeta)R^-(\zeta)$$

where $\zeta$ is on $\Gamma$. The inverse $R^{-1}(w)$ is required to be Hölder continuous on $\Gamma$, analytic in $A^+$, analytic in $A^-$; and of finite degree $n$ at infinity. If $n > 0$, $p(w)$ is a vector polynomial of degree not exceeding $n-1$. If $n \leq 0$, then $p(w) = 0$. If (6.1) is soluble, then $g(z)$ and $p(w)$ must be such that

$$R(w)\left\{ \int_{\Gamma} \frac{[R^+(z)]^{-1}g(z)dz}{z-w} + p(w) \right\}$$

is analytic in $A^+$, and $A^-$ while it vanishes as $w \to \infty$.

The analysis of (6.1) can be simplified if its coefficients satisfy certain analyticity conditions. In fact, we are going to show that if they satisfy the conditions imposed below, then the problem of solving the barrier equation (6.3) can be avoided in a way that leads directly to a closed formula which provides the solution of (6.1).

Consider the more general equation

$$A(\zeta)\hat{\phi}(\zeta) + \int_{\Gamma} B(x,\zeta)\hat{\phi}(z)dz = f(\zeta)$$

and assume the following:
1. \( B(z,w) \) is an analytic matrix function of either variable when each of \( z \) and \( w \) is in \( \Gamma + \Lambda^+ \).

2. \( A(w) \) is analytic in \( \Gamma + \Lambda^+ \).

3. Neither \( C(\zeta) = A(\zeta) + \pi i B(\zeta, \zeta) \) nor \( D(\zeta) = A(\zeta) - \pi i B(\zeta, \zeta) \) vanishes on \( \Gamma \).

4. \( B(\zeta, \zeta) \neq 0 \) for \( \zeta \) on \( \Gamma \).

The vector function

\[
\chi(w) = \int_{\Gamma} \frac{B(z,w)\phi(z)dz}{z-w}
\]

is analytic in \( \Gamma + \Lambda^+ \) and its positive limit value is

\[
\chi^+(\zeta) = \pi i B(\zeta, \zeta)\phi(\zeta) + \int_{\Gamma} \frac{B(z,\zeta)\phi(z)dz}{z-\zeta}
\]

\[
= \pi i B(\zeta, \zeta)\phi(\zeta) - A(\zeta)\phi(\zeta) + f(\zeta)
\]

\[
= -D(\zeta)\phi(\zeta) + f(\zeta) .
\]

From this, \( \phi(\zeta) \) is

\[
\phi(\zeta) = -D^{-1}(\zeta)\chi^+(\zeta) + D^{-1}(\zeta)f(\zeta) .
\]

The substitution of the right-hand side of (6.7) for \( \phi(\zeta) \) in the integral of (6.4) gives
Let $\Lambda^+$ be a domain in $\Lambda^+$ which contains the poles of $D^{-1}(z)$ and $C^{-1}(z)$. Let the counterclockwise periphery of $\Lambda^+$ be $\Gamma$. The deformation of $\Gamma$ in the first integral of (6.8) into $\Gamma$ produces

$$\int_{\Gamma_0} \frac{B(z,\zeta)D^{-1}(z)f(z)dz}{z-\zeta} = f(\zeta).$$

Hence if we use (6.6) we have

$$\phi(\zeta) = c^{-1}(\zeta)[1 + \pi i B(\zeta,\zeta)D^{-1}(\zeta)]f(\zeta)$$

$$- c^{-1}(\zeta) \int_{\Gamma} \frac{B(z,\zeta)D^{-1}(z)f(z)dz}{z-\zeta}$$

$$+ c^{-1}(\zeta) \int_{\Gamma_0} \frac{B(z,\zeta)D^{-1}(z)f(z)dz}{z-\zeta}$$

where $\zeta$ is on $\Gamma$. The formula (6.10) can also be expressed as

$$\phi(\zeta) = \frac{1}{2}[c^{-1}(\zeta) + D^{-1}(\zeta)]f(\zeta) - c^{-1}(\zeta) \int_{\Gamma} \frac{B(z,\zeta)D^{-1}(z)f(z)dz}{z-\zeta}$$

$$+ c^{-1}(\zeta) \int_{\Gamma_0} \frac{B(z,\zeta)D^{-1}(z)f(z)dz}{z-\zeta}. $$
This is the form of the solution of (6.4). Since $\chi(w)$ is analytic in $\Lambda^+$, the integral along $\Gamma_0$, namely

\begin{equation}
(6.12) \quad \oint_{\Gamma_0} B(z, \zeta) D^{-1}(z) \chi(z) dz
\end{equation}

can be calculated in terms of the prescribed functions. The evaluation of (6.12) is elementary if $A(\zeta)$ and $B(\zeta)$ are constant matrices, or matrices whose components are rational functions. If the components of $D^{-1}(z)$ possess no poles in $\Lambda^+$ the integral (6.12) is zero, and then

\begin{equation}
(6.13) \quad \phi(\zeta) = \frac{1}{2}[C^{-1}(\zeta) + D^{-1}(\zeta)] f(\zeta) - C^{-1}(\zeta) \oint_{\Gamma} B(z, \zeta) D^{-1}(z) f(z) dz \quad \zeta
\end{equation}

For more details about the evaluation of (6.12), and ways in which the above method can be extended see Peters [5]. Note that the above procedure remains applicable if the point at infinity is on $\Gamma$. For example, this is the case for the path which coincides with the real axis.

Let us conclude with a simple illustration. The equation

\begin{equation}
(6.14) \quad a(\zeta) \oint_{\Gamma} \frac{\phi_1(z) dz}{z-\zeta} + \oint_{\Gamma} \frac{a(z) \phi_1(z) dz}{z-\zeta} = f_1(\zeta)
\end{equation}

is the particular one which arises from (6.4) when
\[ \phi(\zeta) = \begin{pmatrix} \phi_1(\zeta) \\ \phi_2(\zeta) \end{pmatrix}, \quad f(\zeta) = \begin{pmatrix} f_1(\zeta) \\ 0 \end{pmatrix}, \]

\[ A(\zeta) = \begin{pmatrix} 0 & 0 \\ a(\zeta) & -1 \end{pmatrix}, \quad B(\zeta) = \begin{pmatrix} a(\zeta) & 1 \\ 0 & 0 \end{pmatrix}, \]

\[ C(\zeta) = \begin{pmatrix} \pi i a(\zeta) & \pi i \\ a(\zeta) & -1 \end{pmatrix}, \quad D(\zeta) = \begin{pmatrix} -\pi i a(\zeta) & -\pi i \\ a(\zeta) & -1 \end{pmatrix}, \]

\[ C^{-1}(\zeta) = \frac{1}{2\pi i a(\zeta)} \begin{pmatrix} 1 & \pi i \\ a(\zeta) & -\pi i a(\zeta) \end{pmatrix}, \quad D^{-1}(\zeta) = \frac{1}{2\pi i a(\zeta)} \begin{pmatrix} -1 & \pi i \\ -a(\zeta) & -\pi i a(\zeta) \end{pmatrix}, \]

\[ C^{-1}(\zeta) + D^{-1}(\zeta) = \frac{1}{2\pi i a(\zeta)} \begin{pmatrix} 0 & 2\pi i \\ 0 & -2\pi i a(\zeta) \end{pmatrix}. \]

Suppose that \( a(z) \) is analytic in \( \Gamma + \Lambda^+ \) and that \( a(z) \) does not vanish in \( \Lambda^+ \). From (6.13), since

\[ [C^{-1}(\zeta) + D^{-1}(\zeta)] f(\zeta) = 0 \]

the solution of (6.14) is

\[ \phi_1(\zeta) = -\frac{1}{2\pi^2 a(\zeta)} \int_{\Gamma} \frac{[a(\zeta) + a(z)] f_1(z) dz}{a(z)(z-\zeta)}. \]

(6.15)
References


Pairs of Cauchy Singular Integral Equations and the Kernel \([b(z) + a(\zeta)]/(z - \zeta)\)

May 1971 Technical Report IMM 389

Arthus S. Peters

May 1971

Distribution of this document is unlimited.

This report presents closed formulas for the solution of

\[
a(\zeta) \int \frac{\phi_1(z)dz}{L(z - \zeta)} + \int \frac{b(z)\phi_1(z)dz}{L(z - \zeta)} = f_1(\zeta)
\]

provided the coefficients satisfy

\[
[s(\zeta) - a(\zeta)][s(\zeta) - b(\zeta)] = [(\zeta - \alpha)(\zeta - \beta)]^2 t^2(\zeta)
\]

where \(s(\zeta)\) and \(t(\zeta)\) are rational functions and \(\kappa\) is either zero or one.
<table>
<thead>
<tr>
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<th>LINK B</th>
<th>LINK C</th>
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